The above proof is simpler than that of Dr. von Schaper in that it requires only two applications of the second part of Desargue's theorem; whereas, the latter requires three applications of the first part of this theorem and five of the second part to complete it.

THE DERIVATIVE OF THE LOGARITHM.

By M. B. PORTER, University of Texas.

That the problem of deriving the logarithm presents pedagogic difficulties is sufficiently evident to any one who turns the pages of the texts on the calculus. A great many of these content themselves with showing that $[1 + (1/n)]^n$, n positive and integral, approaches $\Sigma(1/n!)$ as a limit as n becomes infinite and hence that, if $\log x \doteq \log e$ as $x \doteq e +$, the right hand incremental ratio approaches the limit $(1/x) \cdot \log_e a$ when Δx approaches zero over a certain denumerable point set.[1] Some show that this limit is the same over any point set to the right or left of x, though all assume the continuity of $\log x$. The mechanism of this proof involves the binomial theorem for positive integral exponents, simple convergence tests, and obvious inequalities. The main criticisms that can be urged against such proofs is that they are incomplete, that the binomial theorem has usually been proved by an incomplete induction, and that the proof involves many different steps. The steps are simple in themselves, but after all almost as much is assumed as is proved.

In the first edition of Vallée-Poussin's *Cours* an interesting proof of these results is obtained by means of the elementary inequality $a^{n+1} > 1 + (n+1)(a-1)$, followed by the substitution of $[1 + \omega/(n + 1)] \div [1 + (\omega/n)]$ for a.[2] Here, while the steps are all elementary, the obvious artificiality of the whole process unfits it for elementary instruction; the substitution is one that the student would never invent for himself or remember. On the other hand, the Davis-Hedrick *Calculus,* frankly recognizing the unconvincing character of elaborate proof as well as its incompleteness, for the immature mind of the average beginner, makes a stronger appeal to intuition and thus obtains a greater vividness of effect by a sharper, quicker attack and produces quite as satisfactory a state of mental *bien être* on the part of the youthful and uncritical student as that obtained by the more tiresome process, thus following the safe pedagogic principle that it is not worth while to bother the student with details of proof which he cannot understand or at least whose necessity he does not appreciate.

The question of the continuity of the logarithm can be treated by constructing the values of the logarithm function by the insertion of a series of arithmetic and geometric means—the method used by Briggs in the calculation of his tables.

To many teachers of the calculus it seems desirable to put in the hands of the student a simple outline of a proof, which he can fill in, whereby the existence of

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[1] Osborne's Calculus, revised ed., pp. 9–10.

[2] See Granville's *Calculus*, p. 31, where the proof is reproduced.

the limit $\lim_{n=\infty} [1 + (1/n)]^n$ is demonstrated. Since the points to be established in all such proofs are the same, the only simplification possible is in the manner in which these points are established, and here it seems evident that the greatest simplification will be obtained if this process is identical for all the points involved. The writer has tried with considerable success the following procedure.

Lemma. Applying the first law of the mean twice to $(1 - x)^n$ where n is rational, we have

$$\begin{aligned}(1 - x)^n &= 1 - nx(1 - x_1)^{n-1} \\ &= 1 - nx[1 - (n - 1)x_1(1 - x_2)^{n-2}] \\ &= 1 - nx + n(n - 1)xx_1(1 - x_2)^{n-2}, \qquad x > x_1 > x_2 > 0.\end{aligned}$$

Step 1°.

$$\left(1 + \frac{1}{n + 1}\right)^{n+1} > \left(1 + \frac{1}{n}\right)^n,$$

n rational and positive. To prove this show that

$$\frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} \equiv \left(1 + \frac{1}{n}\right)\left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1} \equiv \left(1 + \frac{1}{n}\right)\left[1 - \frac{1}{(n + 1)^2}\right]^{n+1} > 1$$

by applying the lemma to the last bracket.

Step 2.° Show that $[1 + (1/n)]^n$ does not increase indefinitely with n. Apply the lemma to

$$\left(1 + \frac{1}{n}\right)^{-n/2}$$

and thus show that $[1 + (1/n)]^n < 4$.

Step 3°. Show that

$$\lim_{n=+\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n=+\infty} \left(1 - \frac{1}{n}\right)^{-n}$$

by applying the lemma to

$$\frac{\left(1 + \frac{1}{n}\right)^n}{\left(1 - \frac{1}{n}\right)^{-n}} \equiv \left(1 - \frac{1}{n^2}\right)^n.$$

The theorem has now been proved for n rational and the remainder of the proof is filled in as usual by considering the inequality

$$\left(1 + \frac{1}{n + 1}\right)^n < \left(1 + \frac{1}{\omega}\right)^\omega < \left(1 + \frac{1}{n}\right)^{n+1}, \qquad n < \omega < n + 1.$$

We can now calculate e by a table of logarithms.

A proof of this character should, of course, not be given until the use of the mean value theorem has been freely illustrated by applications to particular

functions and numerical problems. Such applications are numerous and interesting. We merely note here such as

$$\begin{array}{lll} \sin x = x - \epsilon & \text{where} & \epsilon < x^3, \\ \cos x - 1 = -(x^2/2) + \epsilon & \text{where} & \epsilon < x^4/4, \qquad x < 1, \\ (1 + x)^{1/n} = 1 + (x/n) - \epsilon & \text{where} & \epsilon < x^2/n, \\ \tan^{-1} x = x - \epsilon & \text{where} & \epsilon < x^3, \end{array}$$

etc., and the justification of the ordinary rules of interpolation in tables of natural sines, cosines, etc., by means of the double application of this theorem. Other applications to maxima and minima problems, and asymptotes at once suggest themselves, but it is not worth while to enter into further detail.

NEW BOOKS RECEIVED.

Solid Geometry. By William Betz and Harrison E. Webb. Ginn and Company, Boston, 1916. xxii + 178 pages. $0.75.

Solid Geometry. By John H. Williams and Kenneth P. Williams. Lyons and Carnahan, Chicago, 1916. xii + 162 pages. $0.80.

Text-Book of Mechanics, Volume VI. Thermodynamics. By Louis A. Martin, Jr. John Wiley and Sons, New York, 1916. xviii + 313 pages. $1.75.

A Community Arithmetic. By Brenelle Hunt. American Book Company, New York, 1916. vii + 277 pages. $0.60.

Analytic Geometry. By W. A. Wilson and J. I. Tracey. D. C. Heath and Company, Boston, 1915. ix + 212 pages. $1.20.

Theory of Errors and Least Squares. A Textbook for College Students and Research Workers. By LeRoy D. Weld. The Macmillan Company, New York, 1916. xii + 190 pages. $1.25.

Goursat's Mathematical Analysis, Volume II, Part I. Functions of a Complex Variable. By E. R. Hedrick and Otto Dunkel. Ginn & Company, Boston, 1916. x + 259 pages. $2.75.

Five-Figure Mathematical Tables. By E. Chappell. The D. Van Nostrand Company, London, 1915. xvi + 320 pages. $2.00.

BOOK REVIEWS.

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Grundriss der Differential- und Integral-Rechnung. L. Kiepert. I Teil: *Differential-Rechnung.* Zehnte Auflage des gleichnamigen Leitfadens von Dr. Max Stegemann. 1905. II Teil: *Integral-Rechnung.* Neunte Auflage, 1908. Helwingsche Verlagsbuchhandlung. Hannover.

This treatise is probably the most popular text on the calculus now available in Germany. The many merits of the work have been preserved in the new edi-